

Survey Comments:

- Outside Resources for assisting homework is ok.
But do not have them do your homework.
 - Any AI usage must be cited.
- Homework grading:
 - I pick a couple of problems to be graded.
 - Rest is completion.
- Reading is not required, but it can be useful before the lecture.
- I enjoyed the fun facts!

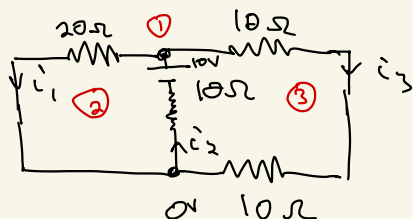
Monday, September 8, Lecture 5: Systems of Linear Equns

Last time...

- Series unit complete! Will return in a few weeks.

A guiding example: Solving for currents with Kirchoff's law

- Goal: find currents i_1, i_2, i_3



Physical information

→ KCL: $\sum \text{incoming currents} = \sum \text{outgoing currents}$

① $\Rightarrow i_1 - i_2 + i_3 = 0$

→ KVL: $\sum_{\text{loop}} \text{voltage} = 0$ + Ohm's law: $V = IR$

② $\Rightarrow$ (left) $20i_1 + 10i_2 + 10 = 0$

③ $\Rightarrow$ (right) $10i_3 + (10+10)i_2 + 10 = 0$

$\Rightarrow$

$$\begin{matrix} \text{①} \\ \text{②} \\ \text{③} \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 20 & 10 & 0 \\ 0 & 10 & 20 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ -10 \end{bmatrix}$$

$A \quad x = b$

This is something a computer can easily solve!

But we should do it ourselves: it will reveal a great deal of structure.

Gaussian elimination:

We first encoded a linear system of eqns. into a matrix. Now, we may algorithmically find it.

. It may be that a system has one, no, or many solution.

- Let's write the "augmented matrix"

$$[A : b]$$

Row Reduction Rules:

i) Interchange two rows

ii) Multiply a row by a nonzero constant

iii) Add a multiple of a row to another

① $\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 20 & 10 & 0 & : & -10 \\ 0 & 10 & 20 & : & -10 \end{bmatrix}$ $\xleftrightarrow{\begin{matrix} \textcircled{2} \rightarrow \frac{1}{10} \textcircled{2} \\ \textcircled{3} \rightarrow \frac{1}{10} \textcircled{3} \end{matrix}}$ $\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 2 & 1 & 0 & : & -1 \\ 0 & 1 & 2 & : & -1 \end{bmatrix}$

$\xleftrightarrow{\textcircled{2} \rightarrow -2 \textcircled{1} + \textcircled{2}}$ $\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 3 & -2 & : & -1 \\ 0 & 1 & 2 & : & -1 \end{bmatrix}$

$\xleftrightarrow{\begin{matrix} \textcircled{2} \leftrightarrow \textcircled{3} \\ \text{swap} \end{matrix}}$ $\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 1 & 2 & : & -1 \\ 0 & 3 & -2 & : & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 1 & 2 & : & -1 \\ 0 & 3 & -2 & : & -1 \end{bmatrix} \xleftrightarrow{\substack{\textcircled{3} \leftarrow -3\textcircled{2} + \textcircled{3} \\ 1\textcircled{3}}} \begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 1 & 2 & : & -1 \\ 0 & 0 & -8 & : & 2 \end{bmatrix}$$

③ $4\frac{1}{4}$ ③

$$24 - 25 + 2$$

$\longleftrightarrow$

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 0 \\ 0 & 1 & 2 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & \frac{1}{4} \end{bmatrix}$$

This is called Reduced Row Echelon Form!

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} \end{bmatrix}$$

Moral:

- use physical law to write equations.
- when they are linear, we can do Row reduction to find an answer.

Solution Structure:

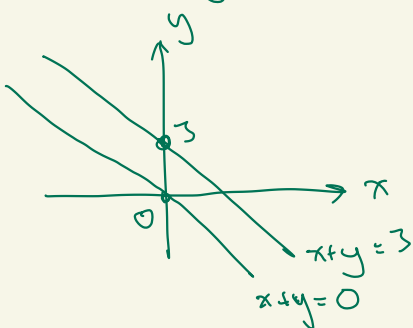
Think about systems of linear equations with x, y .

They can have...

. No solution:

Ex: $x + y = 0$

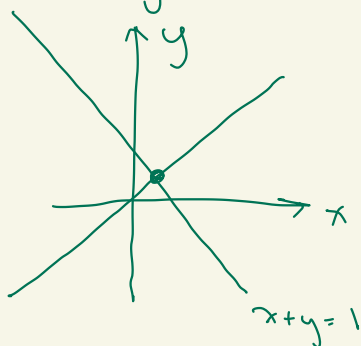
$$x + y = 3$$



. One Soln:

$$x + y = 1$$

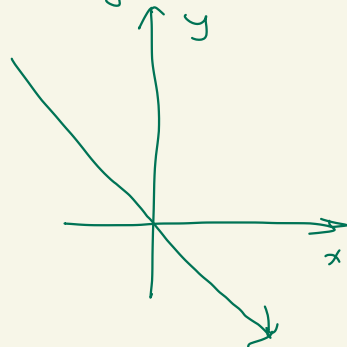
$$x - 2y = 0$$



∞ -Sols:

$$x + y = 0$$

$$2(x + y) = 0$$



Given a system of linear equations, we may always encode it in a matrix:

Let $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\vec{b} = (b_1, \dots, b_m) \in \mathbb{R}^m$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

m equations
 $(\mathbb{R}^m)$

n unknowns

$$\begin{matrix} m \times n \\ \text{matrix} \end{matrix} \rightarrow \begin{matrix} \vec{A} \vec{x} = \vec{b} \\ \begin{matrix} n \times 1 & n \times 1 \end{matrix} \end{matrix}$$

RREF

- Using row operations, every $m \times n$ matrix may be brought to a unique reduced row echelon form.

① First nonzero entry of each row is a 1.

These are called "pivots".

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$
pivot 1 pivot 2 pivot 3

② Pivots are ordered so that each successive pivot is down and to the right:

Nonex:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\uparrow \quad \uparrow$

not RREF!

③ Pivot columns are all 0 except the pivot itself:

Nonex:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

④ All zero rows are at the bottom.

Fact:

Given a matrix equation $A\vec{x} = \vec{b}$, we can read off solutions from

$$\text{ref}([A; \vec{b}])$$

$\uparrow$
Augmented matrix

No solution:

One Soln:

∞ -Sols:

Ex:

$$x + y = 0$$

$$x + y = 1$$

$$x + y = 0$$

$$x + y = 3$$

$$x - 2y = 0$$

$$2(x + y) = 0$$

$[A:b]$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$$

$\text{ref}([A:b])$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

↑
No solution since this means

$$x_1 + x_2 = 0$$

$$0 = 1$$

• inconsistent system

↑
Solution is

$$x_1 = 2/3$$

$$x_2 = 1/3$$

• exactly 1 soln.

↑
Solutions to

$$x_1 + x_2 = 0$$

form a 1 dimensional

line. Any soln.

is given by

choosing x_2 ,

which forces $x_1 = -x_2$

(eg $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a

solution)

Call x_2 a free

variable.

• overdetermined system

intimately related to
range and null space of A !



The RREF contains a lot of information!